9.5 Alternating Series

- Use the Alternating Series Test to determine whether an infinite series converges.
- Use the Alternating Series Remainder to approximate the sum of an alternating series.
- Classify a convergent series as absolutely or conditionally convergent.
- Rearrange an infinite series to obtain a different sum.

Alternating Series

So far, most series you have dealt with have had positive terms. In this section and the next section, you will study series that contain both positive and negative terms. The simplest such series is an **alternating series,** whose terms alternate in sign. For example, the geometric series

$$\begin{aligned}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}\\ &= 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots\end{aligned}$$

is an *alternating geometric series* with $r = -\frac{1}{2}$. Alternating series occur in two ways: either the odd terms are negative or the even terms are negative.

THEOREM 9.14 Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

converge when the two conditions listed below are met.

1. $\lim\limits_{n\to\infty} a_n = 0$

2. $a_{n+1} \le a_n$, for all n

REMARK The second condition in the Alternating Series Test can be modified to require only that $0 < a_{n+1} \le a_n$ for all n greater than some integer N.

Proof Consider the alternating series $\Sigma(-1)^{n+1} a_n$. For this series, the partial sum (where $2n$ is even)

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{2n-1} - a_{2n})$$

has all nonnegative terms, and therefore $\{S_{2n}\}$ is a nondecreasing sequence. But you can also write

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

which implies that $S_{2n} \le a_1$ for every integer n. So, $\{S_{2n}\}$ is a bounded, nondecreasing sequence that converges to some value L. Because $S_{2n-1} - a_{2n} = S_{2n}$ and $a_{2n} \to 0$, you have

$$\begin{aligned}\lim_{n\to\infty} S_{2n-1} &= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n}\\ &= L + \lim_{n\to\infty} a_{2n}\\ &= L.\end{aligned}$$

Because both S_{2n} and S_{2n-1} converge to the same limit L, it follows that $\{S_n\}$ also converges to L. Consequently, the given alternating series converges.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

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EXAMPLE 1 Using the Alternating Series Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}.$$

REMARK The series in Example 1 is called the *alternating harmonic series.* More is said about this series in Example 8.

Solution Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. So, the first condition of Theorem 9.14 is satisfied. Also note that the second condition of Theorem 9.14 is satisfied because

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

for all n. So, applying the Alternating Series Test, you can conclude that the series converges.

EXAMPLE 2 Using the Alternating Series Test

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}.$$

Solution To apply the Alternating Series Test, note that, for $n \geq 1$,

$$\begin{aligned} \frac{1}{2} &\leq \frac{n}{n+1} \\ \frac{2^{n-1}}{2^n} &\leq \frac{n}{n+1} \\ (n+1)2^{n-1} &\leq n2^n \\ \frac{n+1}{2^n} &\leq \frac{n}{2^{n-1}}. \end{aligned}$$

So, $a_{n+1} = (n+1)/2^n \leq n/2^{n-1} = a_n$ for all n. Furthermore, by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{x}{2^{x-1}} = \lim_{x\to\infty} \frac{1}{2^{x-1}(\ln 2)} = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{n}{2^{n-1}} = 0.$$

Therefore, by the Alternating Series Test, the series converges.

EXAMPLE 3 When the Alternating Series Test Does Not Apply

REMARK In Example 3(a), remember that whenever a series does not pass the first condition of the Alternating Series Test, you can use the nth-Term Test for Divergence to conclude that the series diverges.

a. The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots$$

passes the second condition of the Alternating Series Test because $a_{n+1} \leq a_n$ for all n. You cannot apply the Alternating Series Test, however, because the series does not pass the first condition. In fact, the series diverges.

b. The alternating series

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \cdots$$

passes the first condition because a_n approaches 0 as $n \to \infty$. You cannot apply the Alternating Series Test, however, because the series does not pass the second condition. To conclude that the series diverges, you can argue that S_{2N} equals the Nth partial sum of the divergent harmonic series. This implies that the sequence of partial sums diverges. So, the series diverges.

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Alternating Series Remainder

For a convergent alternating series, the partial sum S_N can be a useful approximation for the sum S of the series. The error involved in using $S \approx S_N$ is the remainder $R_N = S - S_N$.

THEOREM 9.15 Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \le a_{N+1}.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 4 Approximating the Sum of an Alternating Series

See LarsonCalculus.com for an interactive version of this type of example.

Approximate the sum of the series by its first six terms.

$$\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n!}\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

Solution The series converges by the Alternating Series Test because

$$\frac{1}{(n+1)!} \le \frac{1}{n!} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n!} = 0.$$

The sum of the first six terms is

$$S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144} \approx 0.63194$$

and, by the Alternating Series Remainder, you have

$$|S - S_6| = |R_6| \le a_7 = \frac{1}{5040} \approx 0.0002.$$

So, the sum S lies between $0.63194 - 0.0002$ and $0.63194 + 0.0002$, and you have $0.63174 \le S \le 0.63214$.

TECHNOLOGY Later, using the techniques in Section 9.10, you will be able to show that the series in Example 4 converges to

$$\frac{e-1}{e} \approx 0.63212.$$

(See Section 9.10, Exercise 58.) For now, try using a graphing utility to obtain an approximation of the sum of the series. How many terms do you need to obtain an approximation that is within 0.00001 unit of the actual sum?

EXAMPLE 5 Finding the Number of Terms

Determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

Solution By Theorem 9.15, you know that

$$|R_N| \le a_{N+1} = \frac{1}{(N+1)^4}.$$

For an error of less than 0.001, N must satisfy the inequality $1/(N+1)^4 < 0.001$.

$$\frac{1}{(N+1)^4} < 0.001 \quad \Longrightarrow \quad (N+1)^4 > 1000 \quad \Longrightarrow \quad N > \sqrt[4]{1000} - 1 \approx 4.6$$

So, you will need at least 5 terms. Using 5 terms, the sum is $S \approx S_5 \approx 0.94754$, which has an error of less than 0.001.

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Absolute and Conditional Convergence

Occasionally, a series may have both positive and negative terms and not be an alternating series. For instance, the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

has both positive and negative terms, yet it is not an alternating series. One way to obtain some information about the convergence of this series is to investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|.$$

By direct comparison, you have $|\sin n| \leq 1$ for all n, so

$$\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}, \quad n \geq 1.$$

Therefore, by the Direct Comparison Test, the series $\sum \left|\frac{\sin n}{n^2}\right|$ converges. The next theorem tells you that the original series also converges.

THEOREM 9.16 Absolute Convergence

If the series $\Sigma\, |a_n|$ converges, then the series $\Sigma\, a_n$ also converges.

Proof Because $0 \leq a_n + |a_n| \leq 2|a_n|$ for all n, the series

$$\sum_{n=1}^{\infty} (a_n + |a_n|)$$

converges by comparison with the convergent series

$$\sum_{n=1}^{\infty} 2|a_n|.$$

Furthermore, because $a_n = (a_n + |a_n|) - |a_n|$, you can write

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

where both series on the right converge. So, it follows that $\Sigma\, a_n$ converges.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The converse of Theorem 9.16 is not true. For instance, the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called **conditional.**

Definitions of Absolute and Conditional Convergence

1. The series $\Sigma\, a_n$ is **absolutely convergent** when $\Sigma\, |a_n|$ converges.
2. The series $\Sigma\, a_n$ is **conditionally convergent** when $\Sigma\, a_n$ converges but $\Sigma\, |a_n|$ diverges.

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EXAMPLE 6 Absolute and Conditional Convergence

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} = \frac{0!}{2^0} - \frac{1!}{2^1} + \frac{2!}{2^2} - \frac{3!}{2^3} + \cdot \cdot \cdot$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdot \cdot \cdot$

Solution

a. This is an alternating series, but the Alternating Series Test does not apply because the limit of the nth term is not zero. By the nth-Term Test for Divergence, however, you can conclude that this series diverges.

b. This series can be shown to be convergent by the Alternating Series Test. Moreover, because the p-series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdot \cdot \cdot$$

diverges, the given series is *conditionally* convergent.

EXAMPLE 7 Absolute and Conditional Convergence

Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} = -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} - \cdot \cdot \cdot$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = -\frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \cdot \cdot \cdot$

Solution

a. This is *not* an alternating series (the signs change in pairs). However, note that

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n(n+1)/2}}{3^n}\right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

is a convergent geometric series, with

$$r = \frac{1}{3}.$$

Consequently, by Theorem 9.16, you can conclude that the given series is *absolutely* convergent (and therefore convergent).

b. In this case, the Alternating Series Test indicates that the series converges. However, the series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\ln(n+1)}\right| = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdot \cdot \cdot$$

diverges by direct comparison with the terms of the harmonic series. Therefore, the given series is *conditionally* convergent.

FOR FURTHER INFORMATION To read more about the convergence of alternating harmonic series, see the article "Almost Alternating Harmonic Series" by Curtis Feist and Ramin Naimi in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

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Rearrangement of Series

A finite sum such as

$$1 + 3 - 2 + 5 - 4$$

can be rearranged without changing the value of the sum. This is not necessarily true of an infinite series—it depends on whether the series is absolutely convergent or conditionally convergent.

1. If a series is *absolutely convergent,* then its terms can be rearranged in any order without changing the sum of the series.
2. If a series is *conditionally convergent,* then its terms can be rearranged to give a different sum.

The second case is illustrated in Example 8.

EXAMPLE 8 Rearrangement of a Series

FOR FURTHER INFORMATION Georg Friedrich Bernhard Riemann (1826–1866) proved that if Σa_n is conditionally convergent and S is any real number, then the terms of the series can be rearranged to converge to S. For more on this topic, see the article "Riemann's Rearrangement Theorem" by Stewart Galanor in *Mathematics Teacher.* To view this article, go to *MathArticles.com.*

The alternating harmonic series converges to ln 2. That is,

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

(See Exercise 55, Section 9.10.)

Rearrange the series to produce a different sum.

Solution Consider the rearrangement below.

$$\begin{aligned}
&1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \cdots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots\right) \\
&= \frac{1}{2}(\ln 2)
\end{aligned}$$

By rearranging the terms, you obtain a sum that is half the original sum.

Exploration

In Example 8, you learned that the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges to $\ln 2 \approx 0.693$. Rearrangement of the terms of the series produces a different sum, $\frac{1}{2}\ln 2 \approx 0.347$.

In this exploration, you will rearrange the terms of the alternating harmonic series in such a way that two positive terms follow each negative term. That is,

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \cdots.$$

Now calculate the partial sums S_4, S_7, S_{10}, S_{13}, S_{16}, and S_{19}. Then estimate the sum of this series to three decimal places.

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9.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Numerical and Graphical Analysis In Exercises 1–4, explore the Alternating Series Remainder.

(a) Use a graphing utility to find the indicated partial sum S_n and complete the table.

n	1	2	3	4	5	6	7	8	9	10
S_n										

(b) Use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum.

(c) What pattern exists between the plot of the successive points in part (b) relative to the horizontal line representing the sum of the series? Do the distances between the successive points and the horizontal line increase or decrease?

(d) Discuss the relationship between the answers in part (c) and the Alternating Series Remainder as given in Theorem 9.15.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin 1$

Determining Convergence or Divergence In Exercises 5–26, determine the convergence or divergence of the series.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3n+2}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{3^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{e^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}(5n-1)}{4n+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+5}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n} n}{\ln(n+1)}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2+4}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n+1)}{n+1}$

17. $\displaystyle\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi}{2}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos n\pi$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{\sqrt[3]{n}}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1\cdot 3\cdot 5 \cdots (2n-1)}$

24. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{1\cdot 4\cdot 7 \cdots (3n-2)}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n - e^{-n}} = \sum_{n=1}^{\infty} (-1)^{n+1}\operatorname{csch} n$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} (-1)^{n+1}\operatorname{sech} n$

Approximating the Sum of an Alternating Series In Exercises 27–30, approximate the sum of the series by using the first six terms. (See Example 4.)

27. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n} 5}{n!}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{\ln(n+1)}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2}{n^3}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$

Finding the Number of Terms In Exercises 31–36, use Theorem 9.15 to determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$

35. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!}$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!}$

Determining Absolute and Conditional Convergence In Exercises 37–54, determine whether the series converges absolutely or conditionally, or diverges.

37. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2^n}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n+3)}{n+10}$

45. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n}}{n\ln n}$

46. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n} e^{-n^2}$

47. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n} n}{n^3-5}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

49. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!}$

50. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}}{\sqrt{n+4}}$

51. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

52. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\arctan n$

53. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi/2]}{n}$

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WRITING ABOUT CONCEPTS

55. Alternating Series Define an alternating series.

56. Alternating Series Test State the Alternating Series Test.

57. Alternating Series Remainder Give the remainder after N terms of a convergent alternating series.

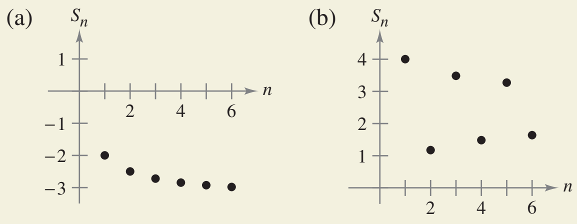
58. Absolute and Conditional Convergence In your own words, state the difference between absolute and conditional convergence of an alternating series.

59. Think About It Do you agree with the following statements? Why or why not?

(a) If both $\sum a_n$ and $\sum(-a_n)$ converge, then $\sum |a_n|$ converges.

(b) If $\sum a_n$ diverges, then $\sum |a_n|$ diverges.

60. HOW DO YOU SEE IT? The graphs of the sequences of partial sums of two series are shown in the figures. Which graph represents the partial sums of an alternating series? Explain.



True or False? **In Exercises 61 and 62, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

61. For the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

the partial sum S_{100} is an overestimate of the sum of the series.

62. If $\sum a_n$ and $\sum b_n$ both converge, then $\sum a_n b_n$ converges.

Finding Values **In Exercises 63 and 64, find the values of p for which the series converges.**

63. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n^p}\right)$ **64.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n+p}\right)$

65. Proof Prove that if $\sum |a_n|$ converges, then $\sum a_n^2$ converges. Is the converse true? If not, give an example that shows it is false.

66. Finding a Series Use the result of Exercise 63 to give an example of an alternating p-series that converges, but whose corresponding p-series diverges.

67. Finding a Series Give an example of a series that demonstrates the statement you proved in Exercise 65.

68. Finding Values Find all values of x for which the series $\sum (x^n/n)$ (a) converges absolutely and (b) converges conditionally.

Using a Series **In Exercises 69 and 70, use the given series.**

(a) Does the series meet the conditions of Theorem 9.14? Explain why or why not.

(b) Does the series converge? If so, what is the sum?

69. $\displaystyle \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \cdots + \frac{1}{2^n} - \frac{1}{3^n} + \cdots$

70. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} a_n,\ a_n = \begin{cases} \dfrac{1}{\sqrt{n}}, & \text{if } n \text{ is odd} \\ \dfrac{1}{n^3}, & \text{if } n \text{ is even} \end{cases}$

Review **In Exercises 71–80, test for convergence or divergence and identify the test used.**

71. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n^{3/2}}$ **72.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+5}$

73. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}$ **74.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

75. $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{7}{8}\right)^n$ **76.** $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$

77. $\displaystyle\sum_{n=1}^{\infty} 100e^{-n/2}$ **78.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4}$

79. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{3n^2-1}$ **80.** $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

81. Describing an Error The following argument, that $0 = 1$, is *incorrect*. Describe the error.

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + \cdots \\ &= 1 \end{aligned}$$

PUTNAM EXAM CHALLENGE

82. Assume as known the (true) fact that the alternating harmonic series

(1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$

is convergent, and denote its sum by s. Rearrange the series (1) as follows:

(2) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots.$

Assume as known the (true) fact that the series (2) is also convergent, and denote its sum by S. Denote by s_k, S_k the kth partial sum of the series (1) and (2), respectively. Prove the following statements.

(i) $S_{3n} = s_{4n} + \frac{1}{2}s_{2n}$, (ii) $S \neq s$

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